ONE AND ONE IS NOTHING:
Liberating Mathematics

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MATHEMATICIANS (and some scientists) tend to feel left out in the cold when they hear reports of adventurous teaching. I have never heard anyone discuss "mathematics and consciousness-raising"; I have not yet heard anyone claim that calculus courses should engage "the whole personality" of either student or teacher; and to date I have not had a colleague attempt to teach "Socratically" in the sense suggested by Rosalyn Sherman.* Yet if one maintains (as I do) that mathematics is a full-fledged member of the liberal arts, one ought to be able to argue that mathematics can be taught in a "liberating" fashion. I believe that this can be done, and note with some joy that mathematics departments in increasing numbers are experimenting in this general direction. The purpose of this essay is to give some rather personal impressions of what mathematics is, of why mathematics is a liberal art, and of what sorts of things might go to make up a "liberating" mathematics course.

While I will mention in passing several types of courses and teaching techniques, my principal interest will be in indicating some topics that might be included in a one- or two-semester course in "pure" mathematics designed for liberal arts students with no prior background in mathematics and no "practical" need for mathematics, e.g., as a prerequisite to their later

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* "Is It Possible to Teach Socratically?", *Soundings,* 53 (1970), pp. 225-245.

studies. I am in effect presupposing a traditional liberal arts curriculum and accepting that many of the students in such a course will be there because it is required for "diversification." I am not here advocating such a context, but simply accepting it as a common context with which we are all familiar. In fact, there may be a built-in advantage in discussing a proposed course in such a context. Many mathematicians feel—and my correspondence seems to confirm—that our nonmathematical colleagues have little idea of what mathematics is, what a mathematician does for research, or what mathematics is doing in the liberal arts. The general public seems to believe that mathematics is to a large degree concerned with numbers, while in fact most research mathematicians can pursue their work for many months without encountering any concrete number larger than two. The situation is further complicated when (as sometimes happens) a mathematician announces in a discussion that pure mathematics is clearly far closer to the fine arts, or even to the humanities, than to science. I hope that by actually discussing some specific pieces of mathematics that might go into a mathematics course for liberal arts students, I can cast some limited amount of light on these issues.

I would like to insert a bit of personal background, by way of illuminating my purpose in thinking of such a course. When I was dangerously young I decided I wanted to become a college professor. Some years later the realization arrived that one must be a professor *of something*; and my interests by that time left me with the initially strange-sounding choice of mathematics or theology. I later found out that this is not an unusual dilemma: a surprising number of mathematicians and theologians started out in the other discipline.* I wound up as a mathematician and a teacher of mathematics in part because I felt that my own religious beliefs and practice would be more of a hindrance than a help to theological teaching and research. This background influences my point of view: I sometimes try to see myself as a college teacher who happens, through circumstance, to be located in a department of mathematics. Of course my writing is not wholly from that viewpoint; I spend my work-

* At that period I was introduced to a number of such people by one of the more prominent ones: the late Bishop Ian Ramsey. His book *Religious Language* (London and New York, 1957) shows a good deal of mathematical influence.

ing days in a department of mathematics and inevitably reflect is prejudices. Yet I would not be teaching mathematics if I did not feel it to be a worthwhile member of the liberal arts and capable of being taught as one. I do believe the liberal arts ought to have something to say about the human condition, the human mind, the human personality; and I very much want to say to my students that thinking about mathematics is not irrelevant to that subject.

Traditional Mathematics Courses

The courses most students are now required to take typically include a semester or two of mathematics; most often this is calculus, probability, or some sort of "business math" which includes algebra, probability, a bit of statistics, and perhaps just a touch of calculus. Now, these courses have their place: they teach a variety of computational techniques that are indispensable at key places in science, engineering, accounting, and even business. They also have numerous disadvantages. One is that at most schools first-year calculus must include all the computational techniques necessary for first-year (and part of second-year) chemistry, physics, engineering, and perhaps psychology or astronomy. The result is that there is very little time to do *mathematics.*

The person whose mathematics at college was confined to calculus probably knows less about what mathematics *is* than the person who gave up after Euclidean geometry in high school. Many mathematics departments regret that their majors must themselves start with calculus, and they usually arrange separate sections or attempt to supplement the usual syllabus in some way. (Unfortunately, even if one could reliably identify mathematics majors at the start of their freshman year, one would still have to enroll them in calculus: the subject matter of mathematics dictates that majors must study real analysis, to which two years of calculus is a prerequisite). It is not even clear that calculus courses as presently constituted do a good job for their natural constituency, physics and engineering students. Certainly they are not the vehicle for teaching the liberal arts student what mathematics has to say to him.

Some other mathematics courses may come closer to the mark. A business mathematics course which contains an unusual amount of linear programming,* or a "finite mathematics" course with a good unit on finite geometry or graph theory, may teach better than calculus the scope of mathematics or the nature of mathematical ideas. At the same time the bulk of the material remains computational, and the tendency for such courses to be considered proper for students "not bright enough" for calculus means that the stress is placed differently than I feel it should be. Computer science courses—for non-technical students—appeal to me for another reason: it is becoming essential in modern society that more people have a grasp of the weaknesses and limitations of computers. But some discussion of this could be included in other courses; it need not occupy a complete course by itself. Finally, a few schools are now experimenting with courses such as "mathematics for ecology" or "mathematics for social action projects." Well done, such programs can make mathematics far more attractive to students than traditional courses, particularly to nontechnical students; but the interest played on here is interest in the applications, rather than in the mathematics as a subject of interest in its own right. I would like to be teaching "pure" mathematics, pure in the sense that, while it may in fact be applicable (in some context, now or in the future), the interest will lie not in the applicability of the work but in its internal beauty or in what it reveals about the power of the human mind.

Before departing from the subject of traditional mathematics courses, some remarks on teaching techniques are in order. First, these materials can be taught inventively in terms of selection of materials and approach. One unit of the customary calculus sequence is devoted to the study of infinite series (for instance, the series 1/2 + 1/4 + 1/8 + 1/6 + ... "adds up" to the number 1, while 1/2 + 1/3 + 1/4 + 1/5 + ... does not "add up"); many techniques are available for deciding whether such a series "adds up," and for finding the total if it does. Students tend to exhibit a moderate interest in what the total is; but they often feel that rules for figuring the total can be applied without regard to whether the series really "adds up."

* Linear programming seems increasingly (and unlike other sorts of mathematics which will occur later in this essay) to be becoming part of the common knowledge used in polite discourse. For instance, note its use at p. 281 of Marna K. and Frederick S. Carney, "The Economics and Ethics of Pollution Control," *Soundings* 54 (1971), pp. 271-287.

Luckily, there is available a film showing a bridge which sways in the wind, getting farther and farther off center until it breaks and falls in the water, and it is not too hard to show students how this results from the false assumption that a certain series "adds up." If the total motion of the bridge "added up," it would add to zero and the bridge would hold; but "on the way" to zero it gets so far off center it collapses. That film, needless to say, is an immense help in teaching calculus.

Second, some interesting thought has been given to what a mathematics course is supposed to accomplish. While traditional calculus courses often seem to contain a fixed collection of "cookbook" techniques, there are at some schools courses which try to teach "problem-solving" without too much restriction of the types of problems to be considered.* In advanced courses for mathematics majors or graduate students the goal may be to teach theorem-proving, and interesting work has been done with what is called by mathematicians the "Moore method." This consists of providing the students with some moderate amount of framework, such as a list of problems to be solved or theorems to be proved, and then allowing them to proceed on their own, with the teacher acting at most as a sort of moderator.

Third, since many courses in mathematics do consist of a list of theorems or problem-solving techniques to be mastered, it is fairly easy to give the student a list and a textbook and let him proceed. This means that a good deal of work has been done with self-paced study, particularly in the mass-enrollment courses such as calculus. It also means that it is easy to separate the teaching and evaluating functions, as discussed by Peter Elbow.† Regrettably, these features do seem to be special to the problem-solving sort of mathematics

* The classic text, very readable, is G. Polya, *How to Solve it* (Princeton, 1945). It is readily available in paperback.

† "Shall We Teach or Give Credit?", *Soundings* 54 (1971), pp. 237-252. For some typical reports on experimental self-paced study, several also involving separation of the teaching and grading functions, see the *Newsletter* of the Committee on the Undergraduate Program in Mathematics, Number 7, February, 1972. For a more detailed exposition of one program, see John Riner, "Individualizing Mathematics Instruction," *American Mathematics Monthly* 79 (January, 1972), pp. 77-86.

course; if in a "liberating" mathematics course we are to have other goals than expertise in problem solving, we are probably forced back toward traditional lectures, discussions, and perhaps even term papers.

Some Mathematical Examples

As stated above, mathematics involves very few numbers. Many elementary essays on mathematics, or elementary books on mathematics, seem to rely heavily on numbers. This may be an attempt to "look like mathematics," or it may be caused by something as innocuous-sounding as typesetting costs: bits of mathematics using few numbers tend to use many diagrams, and diagrams are more expensive to print. But mathematics chiefly involves theorems and proofs; it is more like traditional Euclidean geometry than it is like algebra or trigonometry or even calculus. The "research" done by a mathematics professor in a university customarily consists of discovering and proving new theorems, many thousands of which appear in print annually.*

Hence, a student being subjected to mathematics in the course of a liberal education ought, at the very least, to see enough theorems and proofs to see how they work. In selecting illustrative theorems for lectures I have given with this goal, I have looked for several features. The theorem should answer a question that comes to mind naturally; it may solve a "practical" problem, but it should not be so practical as to be a problem-solving technique rather than a theorem. It should not too closely resemble Euclidean geometry, since much of mathematics does not, and a fair number of the students will have unpleasant memories of high school geometry. It should not involve many numbers. Yet it should identifiably—indisputably—be mathematics rather than simply an exercise in logic. It should require a proof or even be hard to believe until proved, yet the proof once given should be easy, clear, and convincing. These are hard requirements, yet I believe they can be met. In fact, I believe that there is a limited number of theorems

* It is of interest that at almost all universities, the Ph.D. in mathematics is awarded exclusively for the production of new mathematics (new theorems) and not for writing *about* mathematics. This is in contrast to the situation in, e.g., music or poetry where the prospective Ph.D. usually writes a thesis *about* music or poetry rather than composing music or writing poems.

meeting these requirements and yet so easy that they could be introduced in the elementary school. As a first example, I will discuss a theorem I have presented with apparent success to a class of bright third-graders (by apparent success I mean that they enjoyed the theorem and the proof, could carry out the indicated construction, and could even convince another teacher who had not seen the material in advance of the truth of the theorem).

A Theorem on Mazes

Consider an ordinary pencil-line upper-left-to-lower-right maze, such as is shown in Figure 1-A. While one can discuss mathematical ways of solving such a maze, that is not our present goal. Consider the opposite problem, that of composing such mazes. As a child, I dearly loved such mazes and could not get enough; but I was unable to compose them *to fool myself*. I could sketch a "solution," draw a maze around it, and then trace it or erase the solution; but the resulting maze was of limited value to me since I felt I knew the solution in advance. As it turns out, there is a simple method for composing a maze without sketching a solution first; part of the process is shown in Figure 1-B. Draw first the two "outside" parts of the "box,"

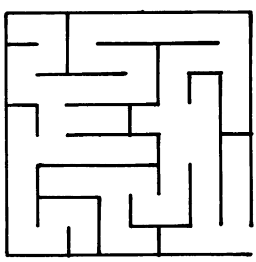

Figure 1-A

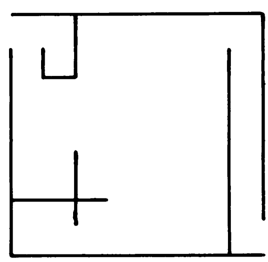

Figure 1-B

leaving the corners open. Then add lines to the inside of the box, following these rules:

(A) Each added line shall be made in one piece, without lifting the pencil; and

(b) Each new line shall touch the previously drawn parts (whether outside lines or added lines) in exactly one place (note that the added lines may make corners or even curve; they may meet earlier lines at an end or may cross them).

Continue in this way until the box is "filled up" to whatever extent you prefer. I contend that the result will always be a maze, and a particularly nice one; and such a contention is the sort of statement eligible to be made into a theorem.

THEOREM. *A maze constructed in this way always has a solution, and in fact no more and no less than one solution.*

If the reader will take the time to draw an example or two, he may decide that this assertion is probably true. Naturally I hope at the same time that he will wonder *why* it is true. One advantage of this theorem is that the proof need not involve so much formality as to hide the "why." Before proceeding with the proof, I ask the reader to take a pencil (or imagination) and draw three lines in Figure 1. First, draw the solution to Figure 1-A. Second, draw in two paths that might be solutions to Figure 1-B *after* the maze was completed, one path going near the top center and the other near the bottom center of the box. We are now ready to proceed with the proof.

Proof. When we drew the box outline (less two corners) the drawing had exactly two pieces. Each time we added a line, it touched exactly one of them; thus the new line did not become a third separate piece, and it did not join the two old pieces into just one piece. Thus the completed maze consists of exactly two pieces. This means that a path can be drawn between the two pieces (i.e., they are really separate), so the maze has a solution. On the other hand, if the maze had two *different* solutions, the two solutions would divide the picture into *more than* two sections (e.g., top, bottom, and center), which is impossible since the completed maze has exactly two pieces.

This is my favorite theorem for elementary school presentation: the pupils invariably agree that it is mathematics, because it requires a clear grasp of the difference between "two" and "not-two"; yet the proof involves no arithmetic. This points up another facet of mathematics: even in those circumstances when numbers do enter, it need not be in a context involving arithmetic.

At this juncture I will introduce a slightly more sophisticated group of results, because they illustrate even more concretely some mathematics which mathematicians find attractive and which is still simple enough to be taught to college freshman (provided they have a moderate background in high school algebra, or can be given one).

Some Theorems on Geodesic Domes

I included in two experimental courses I taught recently units on "combinatorial topology." In this context, "topology" is a sixty-four dollar word for "geometry" and "combinatorial" means "having to do with counting, not with measuring." My first theorem in that unit (which I will ask my reader here to accept on faith) is an old one from Leonhard Euler. Consider a cube: it has six flat sides or faces, twelve edges, and eight corners. A pyramid (with a square bottom, like the Egyptian pyramids) has five faces (four sides and the bottom), eight edges, and five corners. A dodecahedron (one of those desk calendar paperweights with twelve five-sided faces) has twelve faces, thirty edges, and twenty corners. Am I violating my promise to minimize the use of numbers larger than two? Not really, because the point is that in each case the number of corners C, plus the number of faces F, *minus* the number of edges E, is exactly two.

THEOREM (Euler's Formula). *For a solid object with no holes,* $C + F - E = 2$

Proof. The proof is a quite simple counting argument; the reader may find it among other places in George Gamow's paperback *One Two Three . . . Infinity.**

The first time I introduced the above theorem, students were very impressed by the theorem but rather bored by the proof. Fine, it was an unexpected and interesting fact, but a few trials convinced them it was true; why bother to fill the blackboard with drawings supporting a formal argument in its favor? I was hoping to make them happier by going on to the implications of this theorem for possible symmetries of solids, and thus into possible shapes of crystals; but one of those happy coincidences that teachers dream of intervened. When I briefly mentioned the fact that this formula had some applications to geodesic domes, several of the students mentioned the existence of a large geodesic dome model at the far end of the

* New York, 1947. This book has gone through many reprintings and is readily available in paperback; I recommend it heartily to anyone who wants to pick up a little mathematics. The proof of this particular theorem in Gamow's book is reprinted from R. Courant and H. Robbins' book *What is Mathematics?* (London, 1941), which contains far more mathematics but is far slower reading.

campus, which I had never seen. As they described it, it was a full sphere, perhaps eight feet in diameter, made of steel pipe. Considered as a "solid" of the type we are discussing, there were large but unknown numbers of faces, edges, and corners, with only the following concrete facts known:

(A) all of the faces were triangles; and

(B) each corner had five or six edges coming together at it.

The students reported there were "only a few" corners with five edges coming together, most having six edges; several students had tried to count the exact number of corners with five edges, but had lost count at five or six. Those students who had seen the model unanimously disagreed with the statement I wrote on the board:

THEOREM. *If a solid object has no holes, all the faces are triangles, and each corner has either five or six edges coming together at it, then the number of corners with five edges is exactly* twelve.

Proof. Simple algebra will suffice. Picture each edge of the figure as a line having two "ends" and two "sides." Thus if there are E edges, there are 2E ends and 2E sides. We shall count these another way. Each face occupies 3 "sides of edges," so the number of "sides of edges" is also 3F and in fact $3F = 2E$. Similarly, denote the number of 5-edged corners by C_5 and 6-edged corners by C_6; the total number of corners is thus $C = C_5 + C_6$. Since each 5-edged corner consumes 5 "ends of edges" (and similarly for 6) the total number of ends accounted for is $5C_5 + 6C_6$ and we get $2E = 5C_5 + 6C_6$. Now multiplying the known fact $C + F - E = 2$ by 6, we obtain $12 = 6C + 6F - 6E$; we now substitute 4E for 6F, so we can write $12 = 6C + 4E - 6E = 6C - 2E$. Finally we substitute $C_5 + C_6$ for C and $5C_5 + 6C_6$ for 2E, and conclude that

$12 = 6(C_5 + C_6) - (5C_5 + 6C_6) = 6C_5 - 5C_5 + 6C_6 - 6C_6 = C_5$ which is exactly the desired result: $C_5 = 12$, that is, there are 12 corners having 5 edges.

This theorem produced a delightful effect on the class. Almost all of them could follow the individual steps of the proof, but the outcome seemed like "magic"; the result seemed based on insufficient evidence, particularly since most class members had disagreed with the statement of the theorem as first put up. At this point we left the classroom, walked across campus to the geodesic dome in question, and carefully counted the twelve

5-sided corners. From then on, the class not only tended to believe theorems when they believed the proofs, but had a dramatically heightened respect for the power of careful reasoning. This theorem often tempts me to alter so that it might be quoted seriously the oft-quoted dictum of Mark Twain: "There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact."*

One further well-known theorem in this area of mathematics deserves citation while we are on the subject. Techniques similar to the preceding theorem allow us to treat many facts relating to symmetry without any reference to measurement. Let us define a regular solid to be one in which each face has the same number of edges (say r) as every other face, and each corner lies on the same number of edges (say s) as every other corner.

THEOREM. *There are at most five regular solids without holes.*

Proof. Counting "ends of edges" by edges (2 each) and faces (r each) we obtain $2E = rF$. Counting "sides of edges" by edges (2 each) and corners (s each) we obtain $2E = sC$. Substituting $C = 2/sE$ and $F = 2/rE$ in $C + F - E = 2$, and dividing by 2E, yields $1/s + 1/r - 1/2 = 1/E$. Now, s and r must be at least 3 (each face and each corner must involve at least 3 edges) but they cannot both be 4 or more since then $1/s + 1/r - 1/2$ would be less than zero and could not equal 1/E. A bit of trial and error reveals that there are only five possible values for s and r: if $s = 3$, $r = 3$, 4, or 5; if $s = 4$, then $r = 3$; and if $s = 5$, then $r = 3$.

In fact, there are five completely symmetrical regular figures, the tetrahedron (triangular pyramid: $s = r = 3$), cube ($s = 3$, $r = 4$), octahedron (two square pyramids back to back: $s = 4$, $r = 3$), dodecahedron (desk calendar: $s = 3$, $r = 5$), and icosahedron (made of twenty triangles: $s = 5$, $r = 3$). Of course, the proof above gives only a maximum number: it does not guarantee that the figures can actually be built, and says nothing at all about the possibility of, for instance, building them with all edges the same length.

* Quoted in this instance from Clifton Fadiman, *The Mathematical Magpie* (New York, 1962). This book and its predecessor by the same editor, *Fantasia Mathematica* (New York, 1958), are also highly recommended to anyone who wants light reading that gives some notion of the scope and fun of mathematics.

A Mathematics Course for the Liberal Arts Student

Having seen a few pieces of mathematics in some detail, let us turn to the question of what a "liberating" mathematics course might contain. It will contain a healthy number of theorems and proofs, both because they are needed in order to show how mathematics proceeds, and because such mathematics as is done in the course will itself contain them. There are at least two other notions that must be introduced so as not to distort the idea of "what mathematics is": axiom systems, and unsolved problems.

First, the idea of axioms. Before a real problem (e.g., about geodesic domes) can be treated mathematically, one must specify the known conditions ("no holes": our theorems do not apply to doughnuts). Before a collection of mathematical ideas, or a class of real problems, can be treated systematically, one must extract the mathematically important common features. In high school one may get the idea that the axioms or postulates of Euclidean geometry are supposed to be statements about the real world. That, from the mathematician's point of view, is simply not so. One can casually throw away the axiom

For every line and every point not on the line, there is exactly one line through the given point parallel to the given line.

and substitute some other axiom, for instance,

For every line and every point not on the line, there are at least two distinct lines through the given point which do not intersect the given line.

One can then start to prove theorems using this new set of axioms. Of course, they will be different theorems than one has seen before—one of them is that the angles of a triangle always add up to less than 180 degrees—but they are no less interesting mathematically because they do not seem to relate to the real world. At other times, axiom systems are contrived to apply to several different "real-world" situations: since both addition and multiplication are commutative, associative operations (that is, numbers can be combined in any order) any theorem proved about an arbitrary commutative, associative operation will state a true fact about both addition and multiplication: two facts for the price of one. Mathematicians also have a serious interest in the potentialities of axiom systems *per se;* we will return to this later.

Second, we must mention the existence of unsolved problems in mathematics. These are, in effect, statements proposed as theorems for which no proof or disproof has yet been found. One classical example is the Four Color Problem, which loosely reads as follows: suppose one is given a straightforward map of a continent divided into countries. Suppose each country comes in one piece, with no holes and no colonies. One wants to color the map so that whenever two countries have a length of border in common, they are colored with different colors (two countries touching at only one point—like the states Colorado and Arizona—may be given the same color). The following theorem is known:

THEOREM. *Every map may be colored with no more than five colors.* The proof is perfectly accessible to college freshmen; in fact, it follows from Euler's Formula $C + F - E = 2$, with several steps of the proof being very similar to the theorems on geodesic domes above. Unfortunately, in a century of effort, no one has ever found a map that needs five colors. Yet on the other hand, no one has ever been able to prove that four colors are always enough.

Unsolved problems have contributed a great deal to mathematics: whole new branches of mathematics have developed in the attempt to answer them. Yet the unsolved problems are not invariably found in the "outer reaches" of mathematics: some live very close to arithmetic. One such problem is called the Last Theorem of Fermat. Can there be positive whole numbers a, b, c, and n, with n larger than 2, such that an + bn = cn ? (an means a × a × ... × a, n times). It is easy to find such numbers for n = 2:$3^2 + 4^2 = 5^2$. Yet no one has ever found an example for n larger than 2, and no one has ever proved that no such example exists.

This roughly completes a summary of "what mathematics is." If a student gets some notion of the workings of axiom systems, theorems, proofs, and the sorts of unsolved problems that mathematicians like to work on, he will have a good notion of how a mathematician spends his time. If I have been successful so far, the reader (and the student) may agree that the theorems introduced above have an element of beauty, and they may somewhat enlarge one's feeling for the power of logic or the ingenuity of the human mind. Still, if I am to support fairly my claim that mathematics is a liberal art, I ought to offer something more. I want to say that mathematics can be used to say something about problems that occur naturally to the human mind and yet seem "too large" for the human mind to handle in other than a highly speculative manner. Almost by definition, these will be big enough problems that I can barely mention them: yet I will be able to briefly introduce two areas that may be suggestive. One is the notion of infinity; the other is a problem that might be phrased as: "Does every reasonable question have an answer?"

The Notion of Infinity

In view of the fascination that the infinite holds for so many people, it is surprising how few realize that a good deal of mathematical effort has been devoted to the study of "infinity" in its various manifestations. There are at least four or five major ways, and a number of minor ways, in which it has been approached.

We will begin with the notion of "cardinal number" or simply "number." Most traditional undergraduate mathematics courses carefully avoid defining the word *number,* or, if they do define it, do so erroneously. One way to begin which gives a possibility of success is to define first the phrase "the same number as": one pile of things has the same number of things as another pile if we can organize all the things into pairs, with one thing from each pile in each pair. We can now define, if we wish, the phrase "has three things"; a pile has three things in it if it has the same number of things as there are asterisks between the brackets in [* * *]. Now, rather than go on with the rather painful definition of number, suppose that we have given such definitions of "1 thing," "2 things," ..., and let us say (tentatively) that a pile has "infinitely many things" if it has the same number of things as the whole (infinite) list [1, 2, 3, ...]. We are then well on our way to an arithmetic of infinity. We can make (and prove) statements like:

THEOREM. *The list* [1, 2, 3, ...] *has the same number of things in it as the list* [2, 4, 6, ...].

THEOREM. *The collection of all fractions* (1/2, 2/3, 1/12, ...) *has the same number of things in it as the collection of all whole numbers* (1, 2, 3, ...).

THEOREM. *The collection of all points on a line one inch long does* not *have the same number of things in it as the collection of all whole* *numbers: there are more points on the line than there are whole numbers.*

Of course, the last theorem is the exciting one: it implies that there are different sizes even among infinite collections, some larger than others. The reader who would like to see the above theorems written out nicely, and more, is referred to Gamow's *One Two Three . . . Infinity*.

Another somewhat more sophisticated approach to infinity lies in the notion of "ordinal number." When working with cardinal numbers above, we were in effect extending the list 1, 2, 3, ... to include the numbers infinity (all whole numbers), infinity (points on a line), infinity (?), ... It is almost possible to extend the sequence first, second, third, ... to include "infinite numbers"; surprisingly, the infinite numbers in this list look very different from the infinite numbers one gets in the earlier case. Yet a third, and much more sophisticated, approach allows talking about "infinitely small" numbers.

Students have been known to ask which is the "right" approach to infinity. Of course, it is just a matter of different axiom systems, equally useful; but I have encountered certain undergraduate philosopher types so wrapped up in ideas about infinity that they do not like that way out. When all else has failed, I have been known to trot out the parable of the rings from Lessing's *Nathan der Weise*.

In the common mind, the notion of infinity is also tied up with the size and shape of the universe. In a year course—or perhaps in a semester course, with bright enough students or at the cost of other material—one can do enough mathematics to grapple with the notions of a "curved" universe, time as a "fourth dimension," and some similar ideas that can be worked with far more concretely than most people realize.

Does Every Reasonable Question Have an Answer?

In moving to this topic we get into an area called "foundations of mathematics" or even "metamathematics"; it is the province of philosophers as well as mathematicians. Questions tend to be difficult, and the theorems and proofs involved tend to be beyond the ability of most undergraduate mathematics majors, and farther beyond the ability of most freshmen. Nevertheless, there is a result that seems important and suggestive enough that it bears discussing with a larger number of students than now see it. This is the famous Incompleteness Theorem of Kurt Gödel.* Suppose we accept the facts we all know about arithmetic, which form a finite list of facts. In fact, it is possible to write a list of about five axioms from which (with appropriate definitions) all our other knowledge about arithmetic may be deduced as theorems. Now, there are "arithmetic problems" we cannot solve. Fermat's Last Theorem, mentioned above, is one of them. The Four Color Problem is another (while reducing it to a statement about arithmetic may appear difficult, look at our geodesic dome theorems). If there is a map needing five colors, we should be able to find it; if not, we ought to be able to prove there is not. Either there exist or do not exist numbers a, b, c, n, with an + bn = cn; we should be able to find such numbers, or show that no such numbers can be found.

The incompleteness theorem of Gödel, informally stated, says: There exist logical statements, structurally capable of being theorems, which cannot be either proved or disproved simply by applying the rules of logic to our known finite list of facts about arithmetic. Further, this is not just because we have started with the wrong list of facts: for *any* finite collection of axioms for arithmetic, there are statements which cannot be proved or disproved.

Now, it is not known whether Fermat's Last Theorem or the Four Color Problem are among these undecidable statements, and most mathematicians assume they are not. That is, we expect them to be proved or disproved, next month, next year, or perhaps in a hundred years. But it is just possible that a problem mathematicians have tried to solve for years may turn out to be insolvable on the basis of past knowledge: neither true nor false, or rather capable of being called either true or false, either choice creating a new axiom for our list. This happened to us twice in the 1960's: two propositions known as the Continuum Hypothesis and the Axiom of Choice were found to be undecidable.

* One of the more readable formal expositions of this theorem is given in Paul J. Cohen, *Set Theory and the Continuum Hypothesis* (New York, 1966). For a brief and informal exposition, see the article by Nagel and Newman in Kline's book of readings (following note).

Conclusion

A course composed of material selected from that above and similar material from other branches of mathematics might be called "Mathematics Appreciation" or "Mathematics for the Liberal Arts"; at some schools where such a course has been instituted it has been named "An Introduction to Mathematics," probably a less satisfactory title since it is usually a terminal mathematics course. Mathematicians tend to refer to it among themselves as "Mathematics for Poets." When I have taught it, I have tended to do the actual mathematical content in lectures and assign readings of a somewhat more literary nature—e.g., the two mathematical anthologies by Clifton Fadiman, supplemented by (at the most difficult) the Gamow book or others of a comparable level.*

I have now said something about "what mathematics is" and something about the kinds of mathematics that might go into a course for liberal arts students. I hope a few of the topics touched on were attractive enough to provide some support for the claim that mathematics can be aesthetically pleasing. Still, I have not yet explicitly defended my claim that mathematics is a liberal art. I have hoped that the "big questions" raised a few paragraphs ago would be suggestive in this area; absent a clear, applicable definition of liberal arts, it is hard to do more than be suggestive. But it is possible to propose that mathematics reveals something about the human condition.

On the one hand, it shows the human demand for rational system, a craving to generate a multitude of truths from a few basic ones. That is, mathematics expresses the rational passion to analyze complex, frequently unmanageable concepts such as infinity, number, and space in more manageable formal terms. And the imaginative ingenuity, playfulness, and sense of style required of a mathematician are similar to what is required of creative artists in any field.

On the other hand, the content as well as the organization of mathematics may sometimes aid in thinking about the human condition: for instance, Gödel's Incompleteness Theorem shows that there are limits to the ability of mathematics (and perhaps

* The most valuable paperback reference not yet cited is *Readings from Scientific American: Mathematics in the Modern World,* edited by Morris Kline (San Francisco, 1968).

of man) to analyze and systematize. Though mathematical reason has the highest ambitions and an amazing power, there are still undecidable propositions; the situation of the mathematician is not so different from the philosopher's as might have been supposed.

Such observations can emerge naturally and convincingly in courses on mathematics for liberal arts students. My own experience in trying to teach courses based on this sort of content and philosophy has been generally satisfactory. Naturally there have been some difficulties—most conspicuously in finding appropriate material to test or base grades on, since much of it does not promote a particular problem-solving ability of the sort intended by more traditional mathematics courses. Nevertheless the students have responded affirmatively, and I am delighted to see an increasing number of schools with courses of this general type.