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THE TOPOLOGY OF FREE PRODUCTS OF TOPOLOGICAL GROUPS.

by

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1. INTRODUCTION.

In [3], Graev introduced the free product of Hausdorff topological groups G and H (denoted in this paper by $G \amalg H$) and showed it is algebraically the free product G*H and is Hausdorff. While it has been studied subsequently, for example [4,6,7,8,11,12], many questions about its topology remain unsolved. In particular, partial negative results about local compactness were obtained in [7,11,12]. In this paper we obtain a complete solution by showing that $G \amalg H$ is locally compact if and only if G,H and $G \amalg H$ are discrete. A similar line of reasoning allows us to show that $G \amalg H$ has no small subgroups if and only if G and H have no small subgroups.

We are able to obtain much stronger results when G and H are k_ω-spaces, a class of spaces which includes, for example, all compact spaces and all

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connected locally compact groups. In this case we are able to show that the cartesian subgroup, $gp[G,H] = gp\{g^{-1}h^{-1}gh : g \in G,\ h \in H\}$, of $G \amalg H$ is a free topological group, show that certain subgroups of $G \amalg H$ are themselves free products, and show that the topology of $G \amalg H$ depends only on the topologies and not on the algebraic structure of G and H.

2. DEFINITIONS AND PRELIMINARIES.

If X is a completely regular Hausdorff space with distinguished point e, the (Graev) *free topological group on X*, FG(X), is algebraically the free group on $X \setminus \{e\}$, with e as identity element and the finest topology making it into a topological group and inducing the given topology on X; by [2], FG(X) is Hausdorff.

If G and H are topological groups, their *free product* $G \amalg H$ is a topological group whose underlying abstract group is the algebraic free product $G*H$ and whose topology is the finest topology making it into a topological group and inducing the given topologies on G and H; by [3], if G and H are Hausdorff then $G \amalg H$ is Hausdorff.

For the remainder of the paper all topological groups and spaces will be presumed Hausdorff.

A topological group is said to be *NSS* (*or to have no small subgroups*) if there is a neighbourhood of the identity e which contains no subgroup other than {e}. This property is most important for locally compact groups in that Hilbert's fifth problem yields that a locally compact group is a Lie group if and only if it is NSS.

We require the following algebraic preliminaries: The identity map $G \to G$ and the trivial map $H \to \{e\} \subset G$ extend simultaneously to a homomorphism $\pi_1 : G*H \to G$; by [3], this is also a continuous map from $G \amalg H$ to G. Similarly $\pi_2 : G*H \to H$ is a homomorphism and a continuous map on $G \amalg H$. The map $\pi_1 \times \pi_2 : G*H \to G \times H$ has kernel $gp[G,H]$, where $[G,H] = \{g^{-1}h^{-1}gh : g \in G, h \in H\}$. Indeed $gp[G,H]$ is a free group with free basis $[G,H] \setminus \{e\}$. We find it convenient below to introduce a map $c : G \times H \to [G,H]$ given by $c(g,h) = [g,h] = g^{-1}h^{-1}gh$. If w is any element of $G*H$ it has a unique representation $w = ghk$, where $g \in G$, $h \in H$ and $k \in gp\,[G,H]$. We define a map $\pi_c : G*H \to gp[G,H]$ by $\pi_c(w) = k = \pi_2(w)^{-1}\ \pi_1(w)^{-1}\ w$: notice that it is <u>not</u> a homomorphism. Finally we note that there is a bijection (not a homomorphism) $p : G \times H \times gp[G,H] \to G*H$ given by $p(g,h,k) = ghk$. The inverse map is $p^{-1}(w) = (\pi_1(w), \pi_2(w), \pi_c(w))$.

In §4 we use some additional machinery, that of k_ω-spaces; we rely heavily on [4]. A topological space X is said to be a k_ω-space with decomposition $X = \cup X_n$, if $X_1, X_2, \ldots$ are compact subsets of X, $X_n \subset X_{n+1}$ for all n, $X = \bigcup_{n=1}^{\infty} X_n$ and the X_n determine the topology on X in the sense that a subset A of X is closed if and only if $A \cap X_n$ is compact for all n.

The decomposition $X = \cup X_n$ is essential, in that X may be a union of some other ascending chain of compact subsets which fail to determine the topology. If $X = \cup X_n$ and $Y = \cup Y_n$ where where X_n and Y_n are ascending chains of compact sets, the two ascending chains determine the same topology on X provided each X_n is contained in some Y_k and each Y_n is contained in some X_m.

If G is a topological group and a k_ω-space the decomposition $G = \cup G_n$ may be chosen so that the G_n satisfy two additional conditions: if $g \in G_n$ then $g^{-1} \in G_n$, and if $g \in G_n$, $h \in G_k$ then $gh \in G_{n+k}$.

If X is any subset of a group G, we let $gp_n(X)$ denote the set of elements of G which are products of at most n elements of X. Hence $gp_n(G_n) \subset G_{n^2}$.

The class of topological groups which are k_ω-spaces is large enough to include many of the standard examples; in particular, every connected locally compact group is a k_ω-space [12].

We rely heavily on the following result of [4]:

<u>Proposition</u>. Let G be a topological group and X a subset which generates G algebraically. Let $X = \cup X_n$ be a k_ω-space. Then G has the finest group topology consistent with the original topology on X if and only if G is a k_ω-space with decomposition $G = \cup gp_n(X_n)$.

It follows that if $X = \cup X_n$ is a k_ω-space then FG(X) is a k_ω-space with decomposition $FG(X) = \cup gp_n(X_n)$. If $G = \cup G_n$ and $H = \cup H_n$ are k_ω-spaces then $G \quad H$ is a k_ω-space with decomposition $G \quad H = \cup gp_n(G_n \cup H_n)$.

Finally note that when we say that a continuous map $f: X \to Y$ of topological spaces is <u>quotient map</u> we mean that Y has the finest topology for which f is continuous; this is equivalent to requiring that $A \subset Y$ is closed whenever $f^{-1}(A)$ is closed in X.

3. RESULTS FOR GENERAL TOPOLOGICAL GROUPS.

We begin with a few words about Graev's proofs of the existence of free topological groups and free products of topological groups.

Let X be a completely regular space and e a distinguished point of X. Let G(X) be the free group on the set $X\setminus\{e\}$, with e as the identity element of the group. Let $X' = X \cup X^{-1}$. Being completely regular, the topology of X is defined by a family of pseudometrics. Let ρ be a continuous pseudometric on X. Graev extended ρ to a two-sided invariant pseudometric on G(X) as follows: Extend ρ to X' by setting $\rho(x^{-1},y^{-1}) = \rho(x,y)$ and $\rho(x^{-1},y) = \rho(x,y^{-1}) = \rho(x,e) + \rho(y,e)$ for x and y in X. For u and v in G(X) we have an infinity of representations $u = x_1 \ldots x_n$, $v = y_1 \ldots y_n$. where x_i and $y_i \in X'$. Extend ρ to G(X) by setting $\rho(u,v) = \inf\left(\sum_{i=1}^{n} \rho(x_i,y_i)\right)$, where the infimum is taken over all representations $u = x_1 \ldots x_n$ and $v = y_1 \ldots y_n$. The family of all such two sided invariant pseudometrics on G(X) yield a topological group $F_s(X)$. (It is shown elsewhere that $F_s(X)$ is the free topological SIN group on X.) Now $F_s(X)$ is Hausdorff; FG(X) is the group G(X) with the finest Hausdorff topology inducing the original topology on X. This topology FG(X) is in general [10] a finer topology than $F_s(X)$.

Next we let G and H be topological groups. Graev defined a topology τ (not the free product topology, in general) on G*H using the map $p: G \times H \times gp[G,H] \to G*H$. The method requires us to topologize gp[G,H] in some way and then topologize G*H to make the map ρ a homeomorphism. Since p is not a homomorphism it must be checked that this topology τ on G*H is a group topology. (This is in fact quite difficult but our brief comments suppress this difficulty.) Let ρ_G and ρ_H be continuous right invariant pseudometrics on G and H respectively. Define a pseudometric ρ_{GH} on $[G,H]$ by

$$\rho_{GH}(g_1^{-1}g_1^{-1}g_1h_1, g_2^{-1}h_2^{-1}g_2h_2) = \min[\min(\rho_G(g_1,e),\rho_H(h_1,e)) + \min(\rho_G(g_2,e),\rho_H(h_2,e)); \rho_G(g_1,g_2) + \rho_H(h_1,h_2)]$$

The family of all such ρ_{GH} gives rise to a completely regular topology on $[G,H]$. Next, noting that $gp[G,H]$ is a free group on $[G,H]\setminus\{e\}$, we topologize $gp[G,H]$ by putting $(gp[G,H],\tau_1) = F_s[G,H]$. Finally we define the topology τ on $G*H$ by making

$$p: G \times H \times (gp[G,H],\tau_1) \to (G*H,\tau) \text{ a homeomorphism.}$$

Thompson [13] showed that $F_s(X)$ is NSS if and only if X admits a continuous metric. (Thompson's result is stronger than that of Morris and Thompson [9] which showed that $FG(X)$ is NSS if and only if X admits a continuous metric.)

Now if G is NSS, then G admits a continuous metric [9]; so if G and H are NSS, then $G \times H$ admits a continuous metric. Thus $[G,H]$ with the pseudometric topology described above admits a continuous metric. Hence $F_s[G,H]$ is NSS if G and H are NSS. We are now able to prove the following Theorem:

<u>THEOREM 1.</u> <u>$G \amalg H$ is NSS if and only if G and H are NSS.</u>

<u>Proof.</u> If $G \amalg H$ is NSS then any subgroup must be NSS. In particular, G and H must be NSS.

If G and H are NSS, then the above discussion yields that $F_s[G,H]$ is NSS. We shall prove that $(G*H,\tau)$ is NSS, as then G H which has the same algebraic structure but a finer topology will still be NSS. Suppose that $(G*H,\tau)$, which is homeomorphic to $G \times H \times F_s[G,H]$, fails to be NSS. Let N and M be neighbourhoods of e in G and H, respectively, which contain no non-trivial subgroups. Then $\pi_1^{-1}(N) \cap \pi_2^{-1}(M)$ is a neighbourhood of e in $(G*H,\tau)$. Let A be a subgroup contained in $\pi_1^{-1}(N) \cap \pi_2^{-1}(M)$. Since π_1 is a homomorphism and $\pi_1(A) \subset N$ we must have $\pi_1(A) = e$. Similarly $\pi_2(A) = e$. Thus $A \subset F_s[G,H] \subset (G*H,\tau)$. Since $F_s[G,H]$ is NSS, $A = \{e\}$, as desired.

<u>Remarks</u>. (1) This theorem generalizes the main result of [8] which says that if G and H are connected locally compact groups then $G \quad H$ is NSS when and only when G and H are Lie groups.

(2) Note that the proof of Theorem 1 actually yields: $(G*H,\tau)$ is NSS if and only if G and H are NSS.

The fact that $(G*H,\tau)$ is homeomorphic to $G \times H \times gp[G,H]$ leads us to ask if a similar result is true for $G \amalg H$. It is!

<u>THEOREM 2</u>. <u>If $gp[G,H]$ is topologized as a subset of $G \amalg H$, then $G \amalg H$ is homeomorphic to $G \times H \times gp[G,H]$ (the homeomorphism is given by the map p)</u>.

<u>Proof</u>. Since $G \quad H$ is a topological group, the product map $(G \amalg H) \times (G \amalg H) \times (G \amalg H) \to G \amalg H$, given by $(g,h,k) \to ghk$ is continuous, and so is its restriction $p: G \times H \times gp[G,H] \to G \amalg H$. We must show that the inverse map is continuous. The maps $\pi_1: G \amalg H \to G$ and $\pi_2: G \amalg H \to H$ are continuous, so $\pi_c(w) = \pi_2(w)^{-1} \pi_1(w)^{-1} w$ is a product of continuous maps and thus continuous. Hence the map $w \to (\pi_1(w), \pi_2(w), \pi_c(w)) = (g,h,k)$ is continuous, completing the proof.

THEOREM 3. Suppose $G \neq \{e\}$ and $H \neq \{e\}$ are topological groups. Then $G \amalg H$ is not a locally compact space or a complete metric space unless G and H are both discrete. (Of course if G and H are discrete, $G \amalg H$ is also discrete, and consequently locally compact and complete metric.)

Proof. Suppose $G \amalg H$ is a locally compact space or a complete metric space; then so is the closed subgroup $gp[G,H]$. But as $gp[G,H]$ is algebraically a free group it follows from Dudley [1] that $gp[G,H]$ is discrete. Now G is also discrete: for if $\{g_\delta\}$ is a non-constant net converging to $g \in G$ and $h \in H \setminus \{e\}$, then $\{[g_\delta, h]\}$ is a non-constant net converging to $[g,h]$ in $gp[G,H]$, which is impossible. Similarly H is discrete. Finally we see $G \amalg H$, which is homeomorphic to $G \times H \times gp[G,H]$, is also discrete.

Remark. Theorems 2 and 3 hold (with the same proofs) for any group topology μ on $G*H$ for which the projections $\pi_1 : (G*H, \mu) \to G$ and $\pi_2 : (G*H, \mu) \to H$ are continuous and which induce the given topologies on G and H. Thus it would be of interest to answer:

Question 1. Is there any group topology μ on $G*H$ such that either projection $\pi_1 : (G*H, \mu) \to G$ or $\pi_2 : (G*H, \mu) \to H$ is discontinuous?

If continuity of π_1 and π_2 could be shown even under the hypothesis that G, H and $(G*H, \mu)$ are locally compact, we could conclude that no group topology on an algebraic free product is locally compact (except trivially).

What is the topology that $gp[G,H]$ receives as a subset of $G \amalg H$? It is natural to hope that it has a free topological group topology, on an appropriate topology for $[G,H]$.

Question 2. (a) Does the topology induced on $gp[G,H]$ as a subgroup of $G \amalg H$ make it the free topological group $FG[G,H]$?

(b) Is the topology induced on $[G,H]$ as a subset of $G \amalg H$, the same as the quotient topology under the map $G \times H \to [G,H]$ given by $(g,h) \to [g,h]$?

We have already noted that Graev's topology $F_s[G,H]$ is not, in general, $FG[G,H]$. Example 1 in §5 shows that 2(b) is also false for Graev's topology; that is, Graev does not give $[G,H]$ the quotient topology. On the other hand we will answer both 2(a) and 2(b) affirmatively when G and H are k_ω-groups.

4. RESULTS FOR GROUPS WHICH ARE k_ω-SPACES.

We begin by answering Question 2(b) for this case.

THEOREM 4. Let G and H be topological groups which are k_ω-spaces. Then $c: G \times H \to [G,H] \subset G \amalg H$ is a quotient map.

Proof. Let the k_ω-space decompositions of G and H be $G = \cup G_n$ and $H = \cup H_n$. In view of the Proposition stated in §2 $G \amalg H$ is a k_ω-space with decomposition $G \amalg H = \cup gp_n(G_n \cup H_n)$. (Thus a set A is closed in $G \amalg H$ if and only if $A \cap gp_n(G_n \cup H_n)$ is compact for all n, where $gp_n(G_n \cup H_n)$ is the set of elements of $G \amalg H$ which are products of at most n elements of $G_n \cup H_n$; it is compact in $G \amalg H$.)

Now let $A \subset [G,H]$ be such that $c^{-1}(A)$ is closed in $G \times H$. We must show A is closed in $[G,H]$. It will suffice to show A is closed in

$G \amalg H$. We shall prove that $A \cap gp_n(G_n \cup H_n) = c(c^{-1}(A) \cap (G_{n^2} \times H_{n^2})) \cap gp_n(G_n \cup H_n)$; as the right hand side is the intersection of a continuous image of a compact set with a compact set it is compact.

If $n < 4$, both sides are trivial, so assume $n \geq 4$. Now if $w \in gp_n(G_n \cup H_n)$, $w = x_1 \ldots x_n$, with $x_i \in G_n$ or H_n; in reduced form $w = g^{-1}h^{-1}gh$, so clearly g is a product of at most n terms from G_n; hence $g \in G_{n^2}$. Similarly $h \in H_{n^2}$. Since $w = c(g,h)$ we have that $w \in c(c^{-1}(A) \cap (G_{n^2} \times H_{n^2}))$. The other inclusions needed are easy. Hence $A \cap gp_n(G_n \cup H_n)$ is compact for all n, and A is closed, as required.

Note that it follows from the proof of Theorem 4 that $[G,H]$ is closed in $G \amalg H$. We now turn to Question 2(a).

<u>THEOREM 5</u>. <u>Let G and H be topological groups which are k_ω-spaces. Then the topology on $gp[G,H]$ as a subgroup of $G \amalg H$ is the free topological group topology $FG[G,H]$.</u>

<u>Proof</u>. Again let $G = \cup G_n$ and $H = \cup H_n$ be k_ω-space decompositions. Then $G \quad H = \cup gp_n(G_n \cup H_n)$ and $[G,H] = \cup([G,H] \cap gp_n(G_n \cup H_n))$ are k_ω-space decompositions.

Now from the Proposition given in §2, $FG[G,H]$ is a k_ω-space with decomposition

$FG[G,H] = \cup gp_n([G,H] \cap gp_n(G_n \cup H_n))$. On the other hand, $gp[G,H]$ is a closed subgroup of $G \amalg H$ and hence a k_ω-space with decomposition

$gp[G,H] = \cup(gp[G,H] \cap gp_n(G_n \cup H_n))$.

Clearly each $gp_n([G,H] \cap gp_n(G_n \cup H_n))$ is contained in $gp[G,H] \cap gp_k(G_k \cup H_k)$, for $k = n^2$; we must show for each n there is an m such that $gp[G,H] \cap gp_n(G_n \cup H_n) \subset gp_m([G,H] \cap gp_m(G_m \cup H_m))$.

Let $w \in gp[G,H] \cap gp_n(G_n \cup H_n)$. Without loss of generality suppose $n \geq 4$ and write $w = g_1h_2g_3 \cdots g_{n-1}h_n$, each $g_i \in G_n$ and each $h_i \in H_n$. We shall discuss a way of writing w as a product of commutators.

$$\begin{aligned} w &= g_1h_2g_3h_4 \cdots g_{n-1}\, h_n \\ &= [g_1^{-1},h_2^{-1}]\, h_2(g_1g_3)\, h_4 \cdots g_{n-1}\, h_n \\ &= [g_1^{-1},h_2^{-1}][(g_1g_3)^{-1},\, h_2^{-1}]^{-1}(g_1g_3)(h_2h_4)\, g_5 \cdots g_{n-1}\, h_n \\ &= [g_1^{-1},h_2^{-1}][(g_1g_3)^{-1},h_2^{-1}]^{-1}[(g_1g_3)^{-1},(h_2h_4)^{-1}] \cdots (g_1 \cdots g_{n-1})(h_2 \cdots h_n). \end{aligned}$$

The last line has $n - 3$ commutators. Since $\pi_1(w) = \pi_2(w) = e$ we see that $g_1 \cdots g_{n-1} = h_2 \cdots h_n = e$. So w is a product of $n - 3$ commutators $[g,h]^{\pm 1}$, where each g is a product of at most n factors from G_n and hence lies in G_{n^2}. Similarly for h. So for any $m \geq n^2$ we have

$[g,h] \in [G,H] \cap gp_m(G_m \cup H_m)$ and

$w \in gp_m([G,H] \cap gp_m(G_m \cup H_m))$, as desired. Thus the topologies of $FG[G,H]$ and $gp[G,H]$ are the same, completing the proof.

<u>Remark</u>. It follows that if G and H are topological groups and k_ω-spaces, G H contains a free topological group $FG[G,H]$ on a k_ω-space $[G,H]$. In this case we can draw somewhat stronger conclusions than Theorem 3; for instance, $G \amalg H$ is (except trivially) not metrizable and not SIN. (A topological group is said to be a <u>SIN group</u> if every neighbourhood of e contains a neighbourhood of the identity invariant under inner automorphisms of the group.) This leads us to ask

Question 4. If G and H are topological groups, at least one of which is not a discrete space, can $G \amalg H$ be (a) metrizable or (b) a SIN group?

By methods exactly similar to those used in Theorem 5 we obtain

THEOREM 6. Let G and H be topological groups which are k_ω-spaces; let A be a closed subgroup of G and B be a closed subgroup of H. Then the subgroup of $G \amalg H$ generated by $A \cup B$ is closed and is (topologically and algebraically) $A \amalg B$.

For general G and H, A and B closed does imply that the group generated by $A \cup B$ in $G \amalg H$ is closed; this however requires a careful examination of the Graev topology $(G{*}H, \tau)$ introduced before Theorem 1. It does not provide an answer to:

Question 5. Let G and H be topological groups and A and B closed subgroups of G and H respectively. Let $gp(A \cup B)$ denote the subgroup of $G \amalg H$ generated by $A \cup B$. Algebraically it is $A*B$. Is $gp(A \cup B)$ the topological free product $A \amalg B$?

It is natural to ask whether the topology of $G \amalg H$ depends only on the topologies of G and H or also on the group structures. One may be inclined to conjecture that if $f_1 : G_1 \to H_1$ and $f_2 : G_2 \to H_2$ are homeomorphisms, perhaps a homeomorphism $f_1 * f_2 : G_1 * G_2 \to H_1 * H_2$ can be constructed by letting $f_1 * f_2 (r_1 s_1 \ldots r_n s_n) = f_1(r_1)\ f_2(s_1) \ldots f_1(r_n)\ f_2(s_n)$, where $r_i \in G_1$ and $s_i \in G_2$. This fails in general. For instance, if $\{s_\delta\}$ is a net converging to e in G_2, $f_2(e) = e$ and r_1 and r_2 are elements of G_1 with $f_1(r_1)\ f_1(r_2) \neq f_1(r_1 r_2)$, then

$\lim f_1{*}f_2(r_1 s_\delta r_2) = \lim f_1(r_1)\ f_2(s_\delta)\ f_1(r_2) = f_1(r_1)\ f_1(r_2)$

while $f_1{*}f_2(\lim r_1 s_\delta r_2) = f_1{*}f_2(r_1 r_2) = f_1(r_1 r_2) \neq f_1(r_1)\ f_1(r_2)$,

so $f_1{*}f_2$ is discontinuous.

In the k_ω-space case, another approach succeeds:

THEOREM 7. Let G_i and H_i be topological groups which are k_ω-spaces, for $i = 1,2$. If G_i is homeomorphic to H_i, $i = 1, 2$ then $G_1 \amalg G_2$ is homeomorphic to $H_1 \amalg H_2$.

Proof. As $G_1 \amalg G_2$ is homeomorphic to $G_1 \times G_2 \times FG[G_1,G_2]$ and $H_1 \amalg H_2$ is homeomorphic to $H_1 \times H_2 \times FG[H_1,H_2]$ and as $FG(X)$ and $FG(Y)$ are homeomorphic if X and Y are homeomorphic (independent of the choice of basepoints) it will suffice to show that $[G_1,G_2]$ is homeomorphic to $[H_1,H_2]$. Let $f_i : G_i \to H_i$ be a homeomorphism for $i = 1,2$; since topological groups are homogeneous, we may assume that the f_i have been chosen so that $f_i(e) = e$ for each i. Hence the diagram

$$\begin{array}{ccc} G_1 \times G_2 & \xrightarrow{f_1 \times f_2} & H_1 \times H_2 \\ \downarrow c & & \downarrow c \\ [G_1,G_2] & \xrightarrow[j]{} & [H_1,H_2] \end{array}$$

is commutative, where $j\ ([g_1,g_2]) = [f_1(g_1),\ f_2(g_2)]$, and as each vertical map is a quotient map, j is a homeomorphism. This completes the proof.

In view of this it appears that general solutions to Question 2(a) and 2(b) would allow a general solution of:

Question 6. Let G_i and H_i be topological groups for $i = 1,2$. If G_i is homeomorphic to H_i for $i = 1,2$ is $G_1 \amalg G_2$ necessarily homeomorphic to $H_1 \amalg H_2$?

It was shown in Ordman [12] that if G and H are arcwise connected topological groups, then the fundamental group

$$\pi(G \amalg H) = \pi(G \times H) \times L = \pi(G) \times \pi(H) \times L$$

for some group L. It was conjectured that L is always trivial. We now see that $\pi(G \amalg H) = \pi(G) \times \pi(H) \times \pi(gp[G,H])$, where $gp[G,H]$ has the induced topology from $G \quad H$. Further if G and H are k_ω-spaces, then

$$\pi(G \amalg H) = \pi(G) \times \pi(H) \times \pi(FG[G,H]).$$

So the group L has now been identified. However we have been unable to prove that $\pi(FG[G,H])$ is trivial in any case other than the one covered in [12]; that is, when G and H are countable CW-complexes with exactly one zero-cell. It seems reasonable to conjecture that if G and H are simply connected then $\pi(G \amalg H) = \pi(G) \times \pi(H)$. However for this we need to answer

Question 7. If X is simply connected is FG(X) necessarily simply connected? Is it true under the additional assumption that X is a k_ω-space?

5. EXAMPLES.

We conclude by giving two elementary examples which bear on the preceding.

Example 1. The map $c: G \times H \to [G,H] \subset (G*H,\tau)$ is not a quotient map, in general, where τ is Graev's topology. Let $G = H = R$, the additive group of reals with the usual topology. Consider the sequence $a_n = (n, \frac{1}{n})$ in $R \times R$. Now $c(a_n)$ converges to e in $(R*R,\tau)$, for $\rho(c(a_n),e) = \min(|n|,|\frac{1}{n}|) = \frac{1}{n} \to e$, where ρ is the metric (described in §3) arising from the usual metric on each copy of R. However $c(a_n)$ fails to converge to e in $R \amalg R$. To see this note that R is a k_ω-space with decomposition $R = \cup[-n,n]$. Since $\{c(a_k):k = 1,2,\ldots\}$ has finite intersection with each $gp_n([-n,n] \cup [-n,n])$ (here the first $[-n,n] \subset R = G$, the second $[-n,n] \subset R = H$), it is a closed set in $R \amalg R$ and hence does not converge to e.

Since $c(a_n) \in [R,R]$ for all n and $e \in [R,R]$, it follows that $[R,R]$ is topologized differently in $(R{*}R,\tau)$ than in $R \amalg R$. Hence answering Question 2 will require more than an appeal to Graev's topology.

Incidentally the above argument also shows that the topology constructed in Ordman [11(I)] also yields a topology on R*R other than the free product topology.

Example 2. While the free product of compact groups is a k_ω-space, it is very large. Although every discrete subgroup of a compact group is finite, the free product $T \amalg T$ of two circle groups contains a discrete subgroup which is not even finitely generated. Consider the subgroup $\{e,a\}$ of order 2 of the first factor and the subgroup $\{e,b,b^2\}$ of order 3 of the second factor. The free product $\{e,a\} \amalg \{e,b,b^2\}$ is discrete and by Theorem 6 it is a subgroup of $T \amalg T$. Hence its subgroup $gp[\{e,a\},\{e,b,b^2\}]$, the free group on the two generators $x = [a,b]$ and $y = [a,b^2]$ is discrete. This group in turn contains the free group on the countable set $\{x, yxy^{-1}, y^2xy^{-2}, \ldots\}$.

On the other hand, compact subgroups of $T \amalg T$ are very small. Every compact subset of $T \cup T$ is contained in some group $gp_n(T \cup T)$;

that is, has bounded word length. However the only subgroups of $T*T$ with bounded word length are those which are conjugates of subgroups of one of the two factors. Hence every compact subgroup of $T \amalg T$ is either finite, or a conjugate of one of the two factors and hence itself a circle group.

<u>Question 8</u>. What are the locally compact subgroups of $T \amalg T$?

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